

Universal abelian covers of rational surface singularities and multi-index filtrations *

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In [1] and [2], there were computed the Poincaré series of some (multi-index) filtrations on the ring of germs of functions on a rational surface singularity. These Poincaré series were written as the integer parts of certain fractional power series, an interpretation of whom was not given. Here we show that, up to a simple change of variables, these fractional power series are specializations of the equivariant Poincaré series for filtrations on the ring $\mathcal{O}_{\tilde{\mathcal{S}},0}$ of germs of functions on the universal abelian cover $(\tilde{\mathcal{S}},0)$ of the surface $(\mathcal{S},0)$. We compute these equivariant Poincaré series. From another point of view universal abelian covers of rational surface singularities were studied in [6].

Let $(\mathcal{S},0)$ be an isolated complex rational surface singularity and let $\pi : (X, \mathcal{D}) \rightarrow (\mathcal{S},0)$ be a resolution of it (not necessarily the minimal one). Here X is a smooth complex surface, the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$ is a normal crossing divisor on X , all components E_σ ($\sigma \in \Gamma$) of the exceptional divisor \mathcal{D} are isomorphic to the complex projective line \mathbb{CP}^1 and the dual graph of the resolution is a tree.

Let $\mathcal{O}_{\mathcal{S},0}$ be the ring of germs of analytic functions on $(\mathcal{S},0)$. For $\sigma \in \Gamma$, i.e. for a component E_σ of the exceptional divisor, and for $f \in \mathcal{O}_{\mathcal{S},0}$, let $v_\sigma(f)$ be the order of zero of the lifting $f \circ \pi$ of the function f to the space X of the resolution along the component E_σ . Let us choose several components E_1, \dots, E_s of the exceptional divisor \mathcal{D} ($\{1, \dots, s\} \subset \Gamma$). The valuations v_1, \dots, v_s define a multi-index filtration $\{J(\underline{v})\}$ on the ring $\mathcal{O}_{\mathcal{S},0}$: for $\underline{v} = (v_1, \dots, v_s) \in \mathbb{Z}_{\geq 0}^s$, $J(\underline{v}) = \{f \in \mathcal{O}_{\mathcal{S},0} : \underline{v}(f) \geq \underline{v}\}$ (here $\underline{v}(f) = (v_1(f), \dots, v_s(f)) \in \mathbb{Z}_{\geq 0}^s$, $\underline{v}' \geq \underline{v}$

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if and only if $v'_i \geq v_i$ for all $i = 1, \dots, s$). In [1], there was computed the Poincaré series $P(t_1, \dots, t_s)$ of this filtration (the definition of the Poincaré series of a multi-index filtration can be found e.g. in [1, 2, 3]). Let $(E_\sigma \circ E_\delta)$ be the intersection matrix of the components of the exceptional divisor. For $\sigma \neq \delta$, the intersection number $E_\sigma \circ E_\delta$ is equal to 1 if the components E_σ and E_δ intersect (at one point) and is equal to zero if they don't intersect; the self-intersection number $E_\sigma \circ E_\sigma$ of each component E_σ is a negative integer. Let $d = \det(-(E_\sigma \circ E_\delta))$ and let $(m_{\sigma\delta}) = -(E_\sigma \circ E_\delta)^{-1}$. All entries $m_{\sigma\delta}$ are positive and $m_{\sigma\delta} \in (1/d)\mathbb{Z}$. For $\sigma \in \Gamma$, let $\underline{m}_\sigma := (m_{\sigma 1}, \dots, m_{\sigma s}) \in \mathbb{Q}_{\geq 0}^s$.

Let $\overset{\bullet}{E}_\sigma$ be the “smooth part” of the component E_σ in the exceptional divisor \mathcal{D} , i.e., E_σ minus intersection points with all other components of the exceptional divisor \mathcal{D} .

For a fractional power series $S(t_1, \dots, t_s) \in \mathbb{Z}[[t_1^{1/d}, \dots, t_s^{1/d}]]$, let $\text{Int } S(t_1, \dots, t_s)$ be its “integer part”, i.e., the sum of all monomials from $S(t_1, \dots, t_s)$ with integer exponents. In [1] it was shown that

$$P(t_1, \dots, t_s) = \text{Int} \prod_{\sigma \in \Gamma} (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\bullet}{E}_\sigma)}, \quad (1)$$

where $\underline{t}^{\underline{m}} := t_1^{m_1} \cdot \dots \cdot t_s^{m_s}$, $\chi(X)$ is the Euler characteristic of the space X .

A similar formula was obtained in [2] for the Poincaré series of the multi-index filtration on the ring $\mathcal{O}_{\mathcal{S},0}$ defined by orders of a function germ on irreducible components of a curve $(C, 0) \subset (\mathcal{S}, 0)$.

In [1], the fractional power series

$$Q(\underline{t}) = \prod_{\sigma \in \Gamma} (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\bullet}{E}_\sigma)} \quad (2)$$

(and a similar one in [2]) participated as a formal expression convenient to write the formula (1) for the Poincaré series $P(t_1, \dots, t_s)$. There was no interpretation of it.

In [3], there was defined an equivariant Poincaré series for an “equivariant” filtration on the ring $\mathcal{O}_{V,0}$ of germs of functions on a germ $(V, 0)$ of a complex analytic variety with an action of a finite group G . This Poincaré series was computed for a divisorial filtration on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ and for the filtration defined by branches of a G -invariant plane curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$ where the plane \mathbb{C}^2 was equipped with a G -action.

Let $p : (\tilde{\mathcal{S}}, 0) \rightarrow (\mathcal{S}, 0)$ be the universal abelian cover of the surface singularity $(\mathcal{S}, 0)$: see e.g. [5, 6]. One can describe it in the following way. Let $G = H_1(\mathcal{S} \setminus \{0\})$ be the first homology group of the (nonsingular) surface $\mathcal{S} \setminus \{0\}$. The order of the group G is equal to the determinant d of the minus

intersection matrix $-(E_\sigma \circ E_\delta)$ and moreover G is the cokernel $\mathbb{Z}^\Gamma / \text{Im } I$ of the map $I : \mathbb{Z}^\Gamma \rightarrow \mathbb{Z}^\Gamma$ defined by this matrix.

The group G acts on the germ $(\tilde{\mathcal{S}}, 0)$ and the restriction $p|_{\tilde{\mathcal{S}} \setminus \{0\}}$ of the map p to the complement of the origin is a (usual, nonramified) covering $\tilde{\mathcal{S}} \setminus \{0\} \rightarrow \mathcal{S} \setminus \{0\}$ with the structure group G . One can lift the map p to a (ramified) covering $p : (\tilde{X}, \tilde{\mathcal{D}}) \rightarrow (X, \mathcal{D})$ where \tilde{X} is a normal surface (generally speaking not smooth) and $\tilde{X} \setminus \tilde{\mathcal{D}} \cong \tilde{\mathcal{S}} \setminus \{0\}$:

$$\begin{array}{ccc} (\tilde{X}, \tilde{\mathcal{D}}) & \xrightarrow{\tilde{\pi}} & (\tilde{\mathcal{S}}, 0) \\ \downarrow p & & p \downarrow \\ (X, \mathcal{D}) & \xrightarrow{\pi} & (\mathcal{S}, 0) \end{array}$$

(one can define \tilde{X} as the normalization of the fibre product $X \times_{\mathcal{S}} \tilde{\mathcal{S}}$ of the varieties X and $\tilde{\mathcal{S}}$ over \mathcal{S}).

Let g_σ , $\sigma \in \Gamma$ be the element of the group G represented by the loop in $X \setminus \mathcal{D}$ going around the component E_σ in the positive direction. The group G is generated by the elements g_σ for all $\sigma \in \Gamma$. For a point $x \in \dot{E}_\sigma$ and for a point \tilde{x} from the preimage $p^{-1}(x)$ of it, locally, in a neighbourhood of the point \tilde{x} , the map $p : \tilde{D} \rightarrow \mathcal{D}$ is an isomorphism and the map $p : \tilde{X} \rightarrow X$ is a ramified (over \mathcal{D}) covering, the order d_σ of which coincides with the order of the generator g_σ of the group G .

Lemma 1 *The order d_σ of the element $g_\sigma \in G$ is the minimal natural k such that $km_{\delta\sigma}$ is an integer for all $\delta \in \Gamma$.*

Proof. This follows immediately from the fact that $\mathbb{Z}^\Gamma / \text{Im } I \cong \text{Im } m / \mathbb{Z}^\Gamma$ where $m : \mathbb{Z}^\Gamma \rightarrow \mathbb{Q}^\Gamma$ is the map given by the matrix $(m_{\sigma\delta})$ (i.e. minus the inverse of the map I). \square

Let $R(G)$ be the ring of (virtual) representations of the group G . For $\sigma \in \Gamma$, let α_σ be the one-dimensional representation $G \rightarrow \mathbb{C}^* = \mathbf{GL}(1, \mathbb{C})$ of the group G defined by $\alpha_\sigma(g_\delta) = \exp(-2\pi\sqrt{-1}m_{\sigma\delta})$ (here the minus sign reflects the fact that the action of an element $g \in G$ on the ring $\mathcal{O}_{\tilde{\mathcal{S}},0}$ is defined by $(g \cdot f)(x) = f(g^{-1}(x))$).

Let us choose any component \tilde{E}_i of the preimage $p^{-1}(E_i)$ of the component E_i and let \tilde{v}_i be the corresponding divisorial valuation on the ring $\mathcal{O}_{\tilde{\mathcal{S}},0}$. On the space $\bigcup_{\alpha} \mathcal{O}_{\tilde{\mathcal{S}},0}^\alpha$ of all G -equivariant functions on $(\tilde{\mathcal{S}}, 0)$ (α runs over all nonequivalent 1-dimensional representations of the group G) the valuation \tilde{v}_i does not depend on the choice of the component \tilde{E}_i .

In [3], there was defined the equivariant Poincaré series of the multi-index filtration defined by the divisorial valuations $\tilde{v}_1, \dots, \tilde{v}_s$.

Theorem 1 *The equivariant Poincaré series $P^G(t_1, \dots, t_s)$ of the s -index filtration defined by the set of divisorial valuations $\{\tilde{v}_1, \dots, \tilde{v}_s\}$ is given by the formula:*

$$\begin{aligned} P^G(t_1, \dots, t_s) &= \prod_{\sigma \in \Gamma} (1 - \alpha_\sigma \underline{t}^{\underline{d} \underline{m}_\sigma})^{-\chi(\dot{E}_\sigma)} \\ &= \prod_{\sigma \in \Gamma} (1 - \alpha_\sigma t_1^{d_1 m_{1\sigma}} \dots t_s^{d_s m_{s\sigma}})^{-\chi(\dot{E}_\sigma)}. \end{aligned} \quad (3)$$

For a power series $S(t_1, \dots, t_s) = \sum_{\underline{v} \in \mathbb{Z}_{\geq 0}^s} s_{\underline{v}} \underline{t}^{\underline{v}} \in R(G)[[t_1, \dots, t_s]]$ ($R(G)$ is the ring of representations of the group G), let its reduction $\text{red } S(t_1, \dots, t_s)$ be the series $\sum_{\underline{v} \in \mathbb{Z}_{\geq 0}^s} (\dim s_{\underline{v}}) \underline{t}^{\underline{v}} \in \mathbb{Z}[[t_1, \dots, t_s]]$.

Corollary. One has $\text{red } P^G(t_1, \dots, t_s) = Q(t_1^{d_1}, \dots, t_s^{d_s})$, where $Q(\underline{t})$ is the fractional power series defined by (2).

Proof of Theorem 1 For short we shall say that an effective divisor on $\dot{\mathcal{D}} = \bigcup \dot{E}_\sigma$ (or on $\tilde{\mathcal{D}} = p^{-1}(\dot{\mathcal{D}})$) is Cartier if it is the intersection with $\dot{\mathcal{D}}$ (or with $\tilde{\mathcal{D}}$) of the strict transform of a Cartier divisor on $(\mathcal{S}, 0)$ (or on $(\tilde{\mathcal{S}}, 0)$). From [3] it follows that the equivariant Poincaré series $P^G(\underline{t})$ is equal to the integral with respect to the Euler characteristic of the monomial $\alpha \underline{t}^{\underline{v}}$ over the space of G -invariant effective Cartier divisors on $\dot{\mathcal{D}}$. Here $\alpha \in R(G)$ and $\underline{v} \in \mathbb{Z}_{\geq 0}^s$ are functions (in fact semigroup homomorphisms) on the space of G -invariant Cartier divisors on $\dot{\mathcal{D}}$: a G -invariant Cartier divisor defines the orders of zero of the corresponding (G -equivariant) function along the components \dot{E}_i and also the corresponding 1-dimensional representation of the group G .

Thus to compute the equivariant Poincaré series $P^G(\underline{t})$ one has to describe the space of G -invariant effective Cartier divisors on $\tilde{\mathcal{D}}$ and the corresponding functions \underline{v} and α on it.

Lemma 2 *Any G -invariant effective divisor on $\tilde{\mathcal{D}}$ is a Cartier divisor.*

Proof. It is sufficient to show this for the divisor $\sum_{\tilde{x} \in p^{-1}(x)} \tilde{x}$ for a point $x \in \dot{E}_\sigma$,

i.e. for the G -orbit of a point from $\tilde{\dot{E}}_\sigma$. The isotropy group $G_{\tilde{x}}$ of a point $\tilde{x} \in p^{-1}(x)$ is the cyclic subgroup of the group G of order d_σ generated by the element g_σ (this element acts trivially on $p^{-1}(E_\sigma)$).

Let us take the germ at the point x of a smooth curve L_σ on (X, \mathcal{D}) transversal to the exceptional divisor \mathcal{D} . By the Artin criterion (see, e.g., [7], Lemma on page 156), the divisor $d \cdot L_\sigma$ is the strict transform of a Cartier divisor on $(\mathcal{S}, 0)$ (in fact already $d_\sigma L_\sigma$ is one with this property), i.e. there exists a function $f_\sigma : \mathcal{S} \rightarrow \mathbb{C}$ such that the strict transform of the divisor $\{f_\sigma = 0\}$ is $d \cdot L_\sigma$. Let $\bar{f}_\sigma = f_\sigma \circ \pi$ be the lifting of the function f_σ to the space X of the resolution and let $\tilde{f}_\sigma = f_\sigma \circ \pi \circ p$ be the lifting of the function f_σ to the space \tilde{X} of the modification of the universal abelian cover $(\tilde{\mathcal{S}}, 0)$ (\tilde{f}_σ is a G -invariant function on \tilde{X}). Let us describe the divisor $\{\tilde{f}_\sigma = 0\}$. Let $\tilde{L}_{\sigma, \tilde{x}} \subset \tilde{X}$ be the germ at the point $\tilde{x} \in p^{-1}(x)$ of the preimage under the map p of the curve $L_\sigma \subset X$.

The order of zero of the function \tilde{f}_σ along $\tilde{L}_{\sigma, \tilde{x}}$ is equal to d . The order of zero of the function \bar{f}_σ along the component \dot{E}_δ is equal to $d \cdot m_{\sigma\delta}$. The ramification order of the map p over the component \dot{E}_δ is equal to d_δ . Therefore the order of zero of the function $\tilde{f}_\sigma = \bar{f}_\sigma \circ p$ along the preimage of the component \dot{E}_δ is equal to $d \cdot d_\delta \cdot m_{\sigma\delta}$. This (integer) number is divisible by d (since $d_\delta m_{\sigma\delta}$ is an integer: see Lemma 1). Therefore the zero divisor of the function \tilde{f}_σ is divisible by d , i.e. the order of zero of this function along each component of its zero set is divisible by d . This means that a root $\sqrt[d]{\tilde{f}_\sigma}$ of degree d of the function \tilde{f}_σ (i.e. a branch of this root) is well defined up to multiplication by a root of degree d of 1 G -equivariant complex analytic function on \tilde{X} and thus it is the lifting of a G -equivariant function on $(\tilde{\mathcal{S}}, 0)$ (see e.g. [4, page ?]). \square

Corollary. Each G -invariant divisor on the universal abelian cover $(\tilde{\mathcal{S}}, 0)$ of the rational surface singularity $(\mathcal{S}, 0)$ is a Cartier one.

Lemma 2 means that the space of G -invariant effective Cartier divisors on $\tilde{\mathcal{D}}$ is in one to one correspondence with the space of all effective divisors on $\tilde{\mathcal{D}}$. As it follows from the proof of Lemma 2, the order of zero of the G -equivariant function \tilde{f}_σ (corresponding to one point $x \in \dot{E}_\sigma$) along the component \tilde{E}_i is equal to $d_i m_{\sigma i}$. One has to find the (one-dimensional) representation α_σ with respect to which the function \tilde{f}_σ is G -equivariant.

Lemma 3

$$\alpha_\sigma(g_\delta) = \exp(-2\pi\sqrt{-1}m_{\sigma\delta}) .$$

Proof. The element g_δ of the group G acts trivially on the preimage $p^{-1}(\dot{E}_\delta)$ of the component \dot{E}_δ of the exceptional divisor and acts by multiplication by

$\exp(\frac{2\pi}{d_\delta}\sqrt{-1})$ on the normal line to it. The order of zero of the function \tilde{f}_σ along the preimage $p^{-1}(\dot{E}_\delta)$ is equal to $m_{\sigma\delta}d_\delta$. Therefore

$$\frac{g_\delta \cdot f_\sigma}{f_\sigma} = \exp(-\frac{2\pi\sqrt{-1}m_{\sigma\delta}d_\delta}{d_\delta}) = \exp(-2\pi\sqrt{-1}m_{\sigma\delta}) .$$

□

Now Theorem follows from the usual arguments used e.g. in [1, 3]. The space of effective divisors on $\dot{\mathcal{D}} = \bigcup \dot{E}_\sigma$ is the direct product of the spaces of effective divisors on the components \dot{E}_σ . Each of the latter ones is the disjoint union of symmetric powers $S^k \dot{E}_\sigma$ of the component \dot{E}_σ . Therefore

$$P^G(t_1, \dots, t_s) = \prod_{\sigma \in \Gamma} \left(\sum_{k=0}^{\infty} \chi(S^k \dot{E}_\sigma) \cdot \alpha_{\sigma \underline{v}}^{k \underline{d} \underline{m}_\sigma} \right) ,$$

(this follows from the fact that \underline{v} and α are semigroup homomorphisms). The well-known formula

$$\sum_{k=0}^{\infty} \chi(S^k X) t^k = (1-t)^{-\chi(X)}$$

implies the equation (3). □

A similar result holds for the filtration on the ring $\mathcal{O}_{\tilde{\mathcal{S}},0}$ defined by orders of a function germ on branches of a G -invariant curve $(\tilde{C}, 0) \subset (\tilde{\mathcal{S}}, 0)$. Let $\tilde{C} = \bigcup_{i=1}^r \tilde{C}_i$ where \tilde{C}_i are irreducible G -invariant components of the curve \tilde{C} (generally speaking each curve \tilde{C}_i consists of several irreducible components permuted by the group G). Each curve \tilde{C}_i is the preimage under the map p of an irreducible curve C_i on $(\mathcal{S}, 0)$. The curve $\tilde{C} = \bigcup_{i=1}^r \tilde{C}_i$ defines an r -index filtration on the space $\bigcup_{\alpha} \mathcal{O}_{\tilde{\mathcal{S}},0}^{\alpha}$ of G -equivariant functions on the surface $(\tilde{\mathcal{S}}, 0)$ (or on the space $\bigcup_{\alpha} \mathcal{O}_{\tilde{C},0}^{\alpha}$ of G -equivariant functions on the curve $(\tilde{C}, 0)$). Let $\varphi_i : (\mathbb{C}, 0) \rightarrow (\tilde{\mathcal{S}}, 0)$ be a parametrization (uniformization) of an irreducible component of the curve \tilde{C}_i . For a G -equivariant function germ f , let $\tilde{w}_i(f)$ be the order of zero of the function $f \circ \varphi_i$ at the origin: $f \circ \varphi_i(\tau) = a\tau^{\tilde{w}_i(f)} +$ terms of higher degree, $a \neq 0$. The valuations $\tilde{w}_1, \dots, \tilde{w}_r$ define a multi-index filtration in the usual way.

Let $\pi : (X, \mathcal{D}) \rightarrow (\mathcal{S}, 0)$ be a resolution of the surface singularity $(\mathcal{S}, 0)$ which at the same time is an embedded resolution of the curve $(C, 0) \subset (\mathcal{S}, 0)$,

$C = \bigcup_{i=1}^r C_i$. Let \overline{C}_i be the strict transform of the curve C_i in X . Let E_1, \dots, E_s be all the components of the exceptional divisor \mathcal{D} of the resolution. Let $\overset{\circ}{E}_i$ be the “smooth part” of the component E_i in the total transform $\pi^{-1}(C)$ of the curve C , i.e. E_i minus intersection points with all other components of the total transform $\pi^{-1}(C)$. Let $\underline{m}_i = (m_{i1}, \dots, m_{is}) \in \mathbb{Q}_{\geq 0}^s$, $\underline{d} = (d_1, \dots, d_s) \in \mathbb{Z}_{\geq 0}^s$, and a 1-dimensional representation α of the group G ($i = 1, \dots, s$) be defined as above. The same arguments as in the proof of Theorem 1 imply the following statement.

Theorem 2 *The equivariant Poincaré series $P^G(t_1, \dots, t_r)$ of the r -index filtration defined by the set of valuations $\{\tilde{w}_1, \dots, \tilde{w}_r\}$ is given by the formula:*

$$P^G(t_1, \dots, t_r) = \left(\prod_{i=1}^s (1 - \alpha_i \underline{T}^{\underline{d} \underline{m}_i})^{-\chi(\overset{\circ}{E}_i)} \right) \Big|_{T_i \mapsto \prod_{j: \overline{C}_j \cap E_i = pt} t_j}$$

(here $\underline{T} = (T_1, \dots, T_s)$; in the substitution above, $\prod_{j \in \emptyset} t_j$ is supposed to be equal to 1).

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